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Algebraic areas enclosed by Brownian curves on bounded domains

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Abstract. Using constrained path integrals and perturbation theory, we study the algebraic area, $\mathcal{A}$, enclosed by closed Brownian curves on various domains such as rectangles or strips. In the limit of infinitely long curves, the probability distribution $\mathcal{P}(\mathcal{A})$ is shown to be Gaussian. The standard deviation $\langle \mathcal{A}^2 \rangle^{1/2}$ is simply expressed in terms of the length scales of the problem.

Since the pioneering work of Levy [1], the study of the algebraic area, $\mathcal{A}$, enclosed by a closed Brownian curve of length t has aroused great interest. For instance, the problem of the probability distribution $\mathcal{P}(\mathcal{A})$ has been recently re-examined in various contexts [2]. In particular, transport properties in disordered materials in presence of magnetic fields are closely related to $\mathcal{P}(\mathcal{A})$. Knowledge of this distribution allows us to calculate corrections to weak localization (anomalous magnetoresistance [3]) as well as to localization lengths in Anderson insulators [4, 5]. In the case of weak localization, the essential aspect of the physics is indeed governed by interference effects between pairs of time reversed paths. In quasi-two-dimensional samples, the correction to the Drude conductivity is given by [3]

$$\Delta\sigma(B) - \Delta\sigma(0) = \frac{2e^2 D}{\pi\hbar} \left\langle 1 - \cos\frac{2eB\mathcal{A}}{\hbar} \right\rangle$$

where B is the magnetic field and the bracket represents an averaging over all the loops that enclose an algebraic area $\mathcal{A}$. In the case of Anderson insulators, heuristic arguments that account for the effects of the magnetic field on the localization length have been recently proposed [5, 6]. Although the physical picture is quite different, a central role is again played by interference effects. The localization length can be expressed through an average similar to the one mentioned above. When the path of the electron is not restricted by finite sample geometry, this average can be computed with the Levy formula. In finite samples, a plausibility argument involving the central limit theorem [5] permits one to estimate the asymptotic distribution $P(\mathcal{A})$. In this paper, we reconsider this problem using constrained path integrals and elementary perturbation theory. We calculate the asymptotic ($t \to \infty$) distribution $P(\mathcal{A})$ for

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Brownian curves that are restricted to limited regions of the plane such as rectangles, strips or rings. Our analysis corroborates formula (8) of [5].

First, we recall Levy's law for closed Brownian curves of length t and algebraic area $\mathcal{A}$. In the path integral formulation, the probability $\mathcal{P}(\mathcal{A}|\boldsymbol{r}(0))$ for a curve, with initial and final point fixed, to enclose the area $\mathcal{A}$ is

$$\mathcal{P}(\mathcal{A}|\boldsymbol{r}(0)) = N \int_{\boldsymbol{r}(0)=\boldsymbol{r}(t)=\boldsymbol{r}} [\mathcal{D}\boldsymbol{r}] \exp\left(-\int_0^t \dot{\boldsymbol{r}}^2 \, \mathrm{d}t\right) \delta\left(\mathcal{A} - \frac{1}{2}\int_0^t r^2 \dot{\theta} \, \mathrm{d}t\right) \tag{1}$$

(N is a normalization factor; the usual diffusion constant, D, is taken equal to $\frac{1}{2}$).

Using the identity $2\pi\delta(x) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}Bx} \, \mathrm{d}B$, and assuming that the initial (final) point $\boldsymbol{r}(0)$ is not fixed, i.e. that the closed Brownian curve can wander everywhere in the plane, the resulting probability becomes

$$\mathcal{P}(\mathcal{A}) = N' \int_{-\infty}^{+\infty} \mathrm{d}B \, \mathrm{e}^{\mathrm{i}B\mathcal{A}} \, \mathrm{Tr}(\mathrm{e}^{-tH}). \tag{2}$$

The Hamiltonian H appearing in (2) describes the motion of a charged particle in a constant magnetic field:

$$H = \frac{1}{4}\left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}\left(-\mathrm{i}\partial_\theta - \frac{Br^2}{2}\right)^2\right) \tag{3}$$

with the partition function per unit area

$$\mathrm{Tr}(\mathrm{e}^{-tH}) \boxminus Z(B) = \frac{B/4\pi}{\sinh(tB/4)}. \tag{4}$$

The Fourier transformation of $Z(B)$ gives

$$\mathcal{P}(\mathcal{A}) = \frac{\pi}{t} \frac{1}{\cosh^2(2\pi\mathcal{A}/t)}. \tag{5}$$

This is Levy's original result [1]. In particular, it gives a standard deviation that grows like t:

$$\langle\mathcal{A}^2\rangle^{1/2} = t/4\sqrt{3}. \tag{6}$$

Now we consider a Brownian motion on a rectangular domain ($-a/2 \leqslant x \leqslant a/2; -b/2 \leqslant y \leqslant b/2$) with pure reflective boundary conditions (Neumann conditions).

Using for the algebraic area $\mathcal{A}$ the expression

$$\mathcal{A} = \oint(-\alpha y \, \mathrm{d}x + (1-\alpha)x \, \mathrm{d}y) \tag{7}$$

where α is an arbitrary parameter, the Hamiltonian H (equation (3)) now becomes

$$H = \tfrac{1}{4}[(p_x + \alpha By)^2 + (p_y - (1-\alpha)Bx)^2]. \tag{8}$$

This expression shows that α is nothing but a gauge parameter; for the moment we leave it free.

In the asymptotic regime $(t \to +\infty, |\mathcal{A}| \to +\infty)$, we are allowed to write equation (2) as

$$\mathcal{P}(\mathcal{A}) \simeq \int_{-\infty}^{+\infty} \mathrm{d}B\, \mathrm{e}^{\mathrm{i}B\mathcal{A}} \mathrm{e}^{-tE_0(B)} \tag{9}$$

where $E_0(B)$ is the ground state energy of H. Since there are large fluctuations of the phase factor $\mathrm{e}^{\mathrm{i}B\mathcal{A}}$ when $|\mathcal{A}| \to \infty$, only small B values will give significant contributions to $\mathcal{P}(\mathcal{A})$. Thus all we need is the expression of $E_0(B)$ to lowest order in B: our problem is reduced to a simple perturbation calculation.

With Neumann boundary conditions, the unperturbed ($B = 0$) ground state energy (wavefunction) is

$$E_0^{(0)} = 0 \qquad (\psi_0^{(0)} = 1/\sqrt{ab}) \tag{10}$$

and the perturbed energy to lowest order is

$$\Delta E_0 \equiv E_0^{(1)}(\alpha) + E_0^{(2)}(\alpha) \equiv E_0^{(1)}(\alpha) - |E_0^{(2)}(\alpha)|$$

where

$$\begin{aligned} E_0^{(1)}(\alpha) &= \frac{B^2}{48}(\alpha^2 b^2 + (1-\alpha)^2 a^2) \\ |E_0^{(2)}(\alpha)| &= \frac{16B^2}{\pi^6} \\ &\quad \times \sum_{K,K'=0}^{\infty} \frac{(\alpha((2K'+1)^2 b^2 + (2K+1)^2 a^2) - (2K+1)^2 a^2)^2}{((2K'+1)^2 b^2 + (2K+1)^2 a^2)(2K+1)^4 (2K'+1)^4} \end{aligned} \tag{11}$$

ΔE_0 is quadratic in B ($\Delta E_0 = CB^2, C > 0$). Furthermore, we have checked that it does not depend on α. Minimizing $E_0^{(1)}(\alpha)$ with respect to α leads to the gauge

$$\alpha_0 = \frac{a^2}{a^2 + b^2} \equiv \frac{1}{1+\gamma^2} \tag{12}$$

($\gamma = b/a$; we can always choose $\gamma \leqslant 1$), and to the expressions:

$$\begin{aligned} E_0^{(1)}(\alpha_0) &= \frac{B^2}{48}\left(\frac{b^2}{1+\gamma^2}\right) \\ |E_0^{(2)}(\alpha_0)| &= \frac{16B^2}{\pi^6}\frac{b^2}{(1+\gamma^2)^2}F(\gamma) \end{aligned} \tag{13}$$

with

$$\begin{aligned} F(\gamma) &= -\lambda(2)\lambda(4)\gamma^2(1+\gamma^2) + \frac{\pi}{2}\gamma(1+\gamma^2)^2 \\ &\quad \times \left\{ \sum_{K=0}^{\infty} \frac{1}{(2K+1)^5} \left(\coth\left(\frac{(2K+1)\pi}{\gamma}\right) - \frac{1}{2}\coth\left(\frac{(2K+1)\pi}{2\gamma}\right)\right)\right\} \\ &\quad \left(\lambda(n) = \sum_{K=0}^{+\infty} \frac{1}{(2K+1)^n} \right). \end{aligned}$$

For small $\gamma(\leqslant 0.5)$, we have the following approximation:

$$F(\gamma) \simeq -\lambda(2)\lambda(4)\gamma^2(1+\gamma^2) + \frac{\pi}{4}\lambda(5)|\gamma|(1+\gamma^2)^2 \underset{\gamma\to 0}{\longrightarrow} 0. \qquad (14)$$

These expressions are especially interesting when the two lengths a and b are very different. In that case, ΔE_0 is given essentially by $E_0^{(1)}(\alpha_0)$. In contrast, the contribution of $E_0^{(2)}(\alpha_0)$ will be a maximum for the square ($\gamma = 1$).

Coming back to $\mathcal{P}(\mathcal{A})$, we have

$$\mathcal{P}(\mathcal{A}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}B\, \mathrm{e}^{\mathrm{i}B\mathcal{A}} \mathrm{e}^{-t\Delta E_0} \qquad (15)$$

where $\Delta E_0 = CB^2$.

Thus, we get a Gaussian distribution

$$\mathcal{P}(\mathcal{A}) = \frac{1}{2\sqrt{\pi C t}} \mathrm{e}^{-\mathcal{A}^2/4Ct} \qquad (16)$$

the standard deviation of which grows like $\sqrt{t}$:

$$\langle \mathcal{A}^2 \rangle^{1/2} = \sqrt{2Ct}. \qquad (17)$$

As an example, for a square of side length a,

$$\langle A^2 \rangle^{1/2} \simeq \sqrt{t/57}a$$

$$(\langle A^2 \rangle^{1/2} \simeq \sqrt{t/48}a, \text{ if we neglect } E_0^{(2)}(\alpha_0)).$$

Turning to the case of a one-dimensional strip of width b, we see that it suffices to take the limit $a \to \infty$ in these expressions. This leads to

$$\gamma = 0 \qquad \alpha_0 = 1 \qquad E_0^{(1)}(\alpha_0) = \frac{B^2 b^2}{48} \qquad E_0^{(2)}(\alpha_0) = 0.$$

Thus, $\mathcal{P}(\mathcal{A})$ is still a Gaussian of standard deviation

$$\langle \mathcal{A}^2 \rangle^{1/2} = \sqrt{\frac{t}{24}}b. \qquad (18)$$

(In fact, when $a \to \infty$, the unperturbed spectrum becomes continuous. Strictly speaking, we have to consider the low energy states of the Hamiltonian $H = \frac{1}{4}[(p_x + By)^2 + p_y^2]$. The perturbation calculation is straightforward and again gives (18) when $t \to \infty$, $|\mathcal{A}| \to \infty$.) A heuristic derivation of (18) using the central limit theorem can be found in [5].

In figure 1, we compare these results with computer simulations on square lattices. The random walks are limited to (*a*) the whole plane (Levy's law, $a = b = \infty$); (*b*) the strip ($b = 20$, $a = \infty$); (*c*) the square ($a = b = 20$); and (*d*) the rectangle ($b = 20, a = 10$). For each simulation point, we have generated 5000 closed random walks, the number of steps t taking the values 50, 100, 200, ..., 12 800. Theoretical calculations (equations (6), (17), (18)) are represented by the straight lines. At

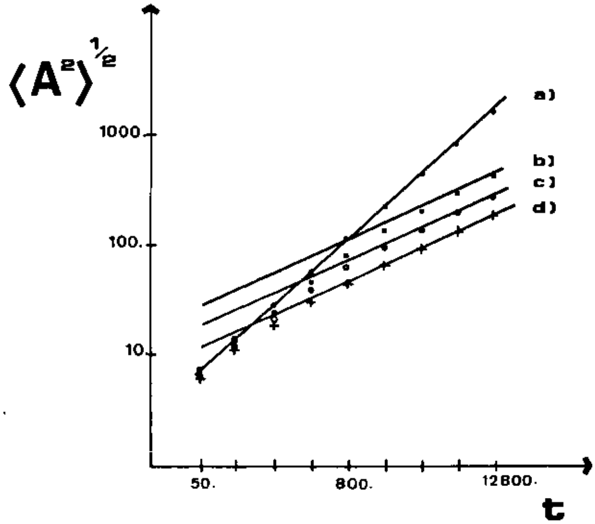

Figure 1. Computer simulations on various two-dimensional square lattices. The width $\langle \mathcal{A}^2 \rangle^{1/2}$ of the distribution $\mathcal{P}(\mathcal{A})$ is plotted (on a log–log scale) as a function of the length t of the curve for the following cases: (*a*) Levy's law (closed circles); (*b*) strip of width $b = 20$ (squares); (*c*) $b = a = 20$ (open circles); (*d*) $b = 20, a = 10$ (crosses). Each point corresponds to 5000 closed random walks. The straight lines represent theoretical results. (For further explanations, see text.)

small t, all the curves follow Levy's law (they do not 'feel' the frontier of the domain). Asymptotically, we observe quite good agreement between the simulations and the theoretical calculations.

We now briefly consider a ring bounded by two concentric circles ($R - b/2 \leqslant r \leqslant R + b/2$ with R the mean radius and b the width, $b \ll R$). This kind of geometry is interesting because of the presence of two length scales. The low energy states are characterized by their angular momentum M

$$E_0(0) \simeq \frac{M^2}{4}\left\langle \frac{1}{r^2} \right\rangle \simeq \frac{M^2}{4R^2}\left(1 + \frac{b^2}{12R^2}\right)$$
$$\psi_0^{(0)} \simeq \frac{1}{\sqrt{2\pi Rb}} e^{iM\theta} \qquad M \in Z \tag{19}$$

and the perturbation is given by

$$E_0^{(1)} = -\frac{BM}{4} + \frac{B^2}{16}\left(R^2 + \frac{b^2}{4}\right)$$
$$E_0^{(2)} = 0. \tag{20}$$

We look at the two limiting cases:

(i) $b^2 \ll t \ll R^2$. All the values of M will contribute to $Z(B)$. This leads to the $\mathcal{P}(\mathcal{A})$ Gaussian with

$$\langle \mathcal{A}^2 \rangle^{1/2} = \sqrt{\frac{t}{24}}\, b. \tag{21}$$

This is precisely the result (18) for the strip. The particle does not have time enough to go round the whole domain.

(ii) $R^2 \ll t$. Only $M = 0$ will contribute to $Z(B)$. The standard deviation becomes

$$\langle \mathcal{A}^2 \rangle^{1/2} = \sqrt{\frac{t}{8}} R \tag{22}$$

a result easily recovered if we consider that, during half of the time, the particle executes a quasi-one-dimensional Brownian motion along the circle of radius R

$$\langle (R\theta)^2 \rangle^{1/2} \simeq \sqrt{\frac{t}{2}} \qquad \langle \mathcal{A}^2 \rangle^{1/2} \simeq \sqrt{\frac{t}{2}} \frac{\pi R^2}{2\pi R} = \sqrt{\frac{t}{8}} R.$$

To conclude, we give the results for a rectangular domain when Dirichlet boundary conditions are used (i.e. we only consider Brownian curves that never meet the boundary). Equations (13) become

$$\begin{aligned} E_0^{(1)}(\alpha_0)_{\mathrm{Dir}} &= E_0^{(1)}(\alpha_0)_{\mathrm{Neum}} \left(1 - \frac{6}{\pi^2}\right) \\ |E_0^{(2)}(\alpha_0)| &= \frac{(128)^2}{\pi^6} B^2 \left(\frac{\gamma}{\gamma^2+1}\right)^2 b^2 \\ &\times \sum_{K,K' \geqslant 1} \left\{ \frac{K 2 K'^2}{(4K^2-1)^2(4K'^2-1)^2} \frac{[(4K'^2-1)^{-1} - (4K^2-1)^{-1}]^2}{(4K'^2-1)\gamma^2 + (4K^2-1)} \right\}. \end{aligned} \tag{23}$$

For the strip, the standard deviation (18) is simply multiplied by a factor $\sqrt{1 - 6\pi^{-2}}$.

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